

## Conditional stability theorem for the one dimensional Klein-Gordon equation

Aslihan Demirkaya<sup>1,2</sup> and Milena Stanislavova<sup>1,2,a)</sup>

<sup>1</sup>*Mathematics Department, University of Hartford, 200 Bloomfield Avenue,  
West Hartford CT 06112, USA*

<sup>2</sup>*Department of Mathematics, University of Kansas, 1460 Jayhawk Boulevard,  
Lawrence KS 66045–7523, USA*

(Received 30 November 2010; accepted 24 October 2011; published online 18 November 2011)

The paper addresses the conditional non-linear stability of the steady state solutions of the one-dimensional Klein-Gordon equation for large time. We explicitly construct the center-stable manifold for the steady state solutions using the modulation method of Soffer and Weinstein and Strichartz type estimates. The main difficulty in the one-dimensional case is that the required decay of the Klein-Gordon semigroup does not follow from Strichartz estimates alone. We resolve this issue by proving an additional weighted decay estimate and further refinement of the function spaces, which allows us to close the argument in spaces with very little time decay. © 2011 American Institute of Physics. [doi:[10.1063/1.3660780](https://doi.org/10.1063/1.3660780)]

### I. INTRODUCTION

In this paper, we are interested in the asymptotic stability of steady state solutions of Klein-Gordon type equations:

$$u_{tt} - \Delta u + u - \mathcal{N}(u) = 0, \quad (t, x) \in \mathbf{R}^+ \times \mathbf{R}^d, \quad (1)$$

where  $\mathcal{N}(u)$  is the nonlinear term. With some assumptions on the nonlinear term, it has been proved by the authors of Ref. 14 that these solutions are in fact linearly and nonlinearly unstable. Our interest is the conditional stability of such steady state solutions. This kind of stability has been extensively studied recently. For example for the equation  $u_{tt} - \Delta u = |u|^5$ , in Ref. 14, the existence of steady state solutions, the linear and the nonlinear instability of such solutions have been proved. However, it has been also proved in Ref. 16 that for the special perturbation to the steady state solution of  $u_{tt} - \Delta u = |u|^5$ , the solution exists globally and remains near the steady state. Thus, a center-stable manifold for the steady state in the sense of Bates and Jones<sup>1</sup> is described. In 1989, Bates and Jones<sup>1,2</sup> proved that for a large class of semilinear equations, including the Klein-Gordon equation, the space of solutions decomposes into an unstable and center-stable manifold. Similar result was proved in Ref. 10 for the semilinear Schrödinger equation in any dimension. Both are abstract results and do not deal with the global in time behavior of the solutions, e.g., existence and asymptotic behavior. The first asymptotic stability result was obtained by Soffer and Weinstein,<sup>22,23</sup> (see also Ref. 24), followed by works of Pillet and Wayne,<sup>20</sup> Buslaev, Perelman, Sulem,<sup>5–7</sup> Rodnianski-Schlag-Soffer<sup>21</sup> etc. In this context, we would like to mention some recent work of Schlag,<sup>25</sup> Krieger and Schlag<sup>15</sup> and Beceanu<sup>3,4</sup> on the existence of center-stable manifold for the pulse solutions of the focusing cubic nonlinear Schrödinger equation in dimension three. It identifies a center-stable manifold in the critical for the equation space  $H^{1/2}$  and shows that solutions starting on the manifold exist globally in time and remain on the manifold for all time answering an open question in Ref. 10. Recently the authors of Ref. 26 proved a conditional stability of the steady state solutions of (1)

<sup>a)</sup> Author to whom correspondence should be addressed. Electronic mail: [stanis@math.ku.edu](mailto:stanis@math.ku.edu).

with  $\mathcal{N}(u) = |u|^{p-1}u$  for the dimension  $d = 2, 3$  and  $4$  where  $p \geq 1 + 4/d$ . In terms of center-stable manifold for the solution, their result shows the global in time behavior of the solutions and a precise description of the manifold which includes its co-dimension and decay rates. In these problems, since Strichartz estimates are key, the lower the dimension, the harder it is to close the argument. The main difficulty in the one-dimensional case is that the required decay of the Klein-Gordon semigroup does not follow from Strichartz estimates alone. One needs to further refine the function spaces and use additional decay estimates to resolve this issue. The techniques we use are similar to the ones used in Refs. 15, 18, and 26.

In this paper, we consider steady state solutions for the equation,

$$u_{tt} - u_{xx} + u - |u|^{p-1}u = 0 \quad (t, x) \in \mathbf{R}^+ \times \mathbf{R}^1, \quad (2)$$

for  $p \geq 5$  and explicitly construct the center-stable manifold for such solutions.

We will introduce some notations that we will use throughout this paper. We denote

$$\begin{aligned} \|f\|_{L_t^r L_x^s(\mathbf{R}; \langle x \rangle^p dx)} &= \left( \int_{\mathbf{R}} \left( \int_{\mathbf{R}} \langle x \rangle^p |f(t, x)|^s dx \right)^{r/s} dt \right)^{1/r} \\ \|f\|_{L_x^s(\mathbf{R}; \langle x \rangle^p dx) L_t^r} &= \left( \int_{\mathbf{R}} \langle x \rangle^p \left( \int_{\mathbf{R}} |f(t, x)|^r dt \right)^{s/r} dx \right)^{1/s} \end{aligned}$$

and  $\langle x \rangle = \sqrt{1 + x^2}$ .

The existence and uniqueness of steady state solutions of (1) are shown in Ref. 17 for  $p < \frac{d+2}{d-2}$  when  $d \geq 3$  and for any  $p$  when  $d = 1, 2$ . These solutions are positive, radial and exponentially decaying. Next lemma in Ref. 8 shows the explicit form of such solutions for (2).

*Lemma 1.1: For all  $p \in (1, \infty)$  the steady state solution  $\phi(x)$  of (2) has the explicit form*

$$\phi(x) = c_p \cosh^{-\beta} \left( \frac{x}{\beta} \right), \quad c_p = \left( \frac{p+1}{2} \right)^{\frac{1}{p-1}}, \quad \beta := \frac{2}{p-1} \quad (3)$$

$\phi(x)$  satisfies (2) and is the unique  $H^1(\mathbf{R})$ -solution up to translation.

The linearization of (2) around the steady state solution  $\phi$  is given by the operator  $\mathcal{H} := -\partial_x^2 + 1 - p\phi^{p-1}$ . The spectral stability of the steady state solutions is determined by the spectrum of the operator  $\mathcal{H}$ . Next lemma gives the spectrum and the corresponding eigenfunctions.

*Lemma 1.2: (See Theorem 3.1 in Ref. 9) For the equation (2), assume  $3 \leq p < \infty$ . Then there exists  $\sigma = \sigma(p) > 0$ , such that the spectrum of  $\mathcal{H}$  is given by*

$$\sigma(\mathcal{H}) = \{-\sigma^2\} \cup \{0\} \cup [1, \infty) \quad (4)$$

with  $\mathcal{H}\psi = -\sigma^2\psi$ . The eigenfunctions  $\{\psi\}$  and  $\{\phi'\}$  (corresponding to the eigenvalue at  $-\sigma^2$  and  $0$  respectively) are decaying at infinity and mutually orthogonal.

In particular, in the one-dimensional case the so called “gap lemma” for the spectrum is satisfied if  $p \geq 3$ , namely there are no eigenvalues in  $(0, 1]$ .

Another issue that will be important in our argument is the absence of resonances. More precisely, we say that a resonance occurs at  $k$ , if there is a bounded solution  $\phi$  to the equation  $\mathcal{H}[\phi] = k\phi$ . We use an equivalent condition, namely a resonance occurs at  $k$ , if the Wronskian  $W(k)$  of the Jost solutions vanishes at  $k$ , see Sec. III C below. This phenomenon presents a well-known complication in establishing dispersive estimates. However, resonances for operators like  $\mathcal{H}$  do not occur anywhere, except possibly at the edge of the continuous spectrum, that is at  $1$ . In the explicit one-dimensional case under consideration, we rule this out. In fact, we have the following result

*Lemma 1.3:* For  $p > 1$ , the operator  $H_p$  has resonance, exactly at  $p_j = 1 + \frac{2}{j-1}$ ,  $j = 2, 3, \dots$ . In particular, for  $p > 3$ ,  $\mathcal{H}_p$  is resonance free.

We prove Lemma 1.3 in the Appendix, but it follows easily from the theory developed in Ref. 28. The resonances at  $p_j$  can also be explained easily from the complete eigenvalue picture provided in Ref. 9. Namely, for each  $p_j$  and  $p$  close to  $p_j$ , we have eigenvalues  $\lambda(p)$ , so that  $\lim_{p \rightarrow p_j} \lambda(p) = 1$ . Thus, as  $p \rightarrow p_j$ , regular eigenvalues collide with the continuous spectrum creating resonances at the bifurcation points  $p_j$ .

Next, we will describe an explicit construction of the center-stable manifold  $\Sigma$ , which is our main result. This conditional stability theorem states that if the initial data  $u_0$  satisfies  $u_0 - \phi \in \Sigma$ , then the solution will approach in an exponential way or slower the steady state  $\phi$ . In this theorem, we will assume the initial data to be even. This will destroy the eigenvalue at 0. Since the evolution preserves even solutions and the zero eigenvalue has only odd eigenfunctions, the whole evolution proceeds perpendicularly to that marginally stable direction. Thus we will be looking for a solution  $u$  in the form (5). More precisely, we will write differential equations for the unknown functions  $a(t)$  and  $z(t)$ , which we will solve using fixed points for certain maps. We will show that these maps do indeed have fixed points, in view of the linear estimates that they satisfy. These will be in turn a consequence of the spectral assumptions and the decay of the bound state. Our main theorem is given next.

**Theorem 1.4:** For (2) with  $5 \leq p < \infty$ , and  $\mathcal{H}\psi = -\sigma^2\psi$  where  $\sigma = \sigma(p)$ , there exists  $0 < \epsilon = \epsilon(p) < 1$  and  $0 < \delta = \delta(p) < 1$ , and a function

$$h : B_{H^1}(\delta\epsilon) \times B_{L^2}(\delta\epsilon) \cap \{(f, g) : \langle \sigma f + g, \psi \rangle = 0\} \rightarrow \mathbf{R}^1$$

so that whenever the real-valued initial data is even and

$$\begin{aligned} u(0) &= \phi + f_1 + h(f_1, f_2)\psi \\ u_t(0) &= f_2 \end{aligned}$$

$$\langle \sigma f_1 + f_2, \psi \rangle = 0; \|(f_1, f_2)\|_{H^1 \times L^2} < \delta\epsilon,$$

then

$$u(t, x) = \phi(x) + a(t)\psi + \mathbf{z}(t, x) \quad (5)$$

$$\mathbf{z} = P_{a.c.}(\mathcal{H})\mathbf{z}.$$

$$\|z\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1 \cap L_x^\infty(\mathbb{R}; \langle x \rangle^{-3/2} dx) L_t^2} \leq \epsilon$$

$$\|a\|_{L_t^3[0, \infty) \cap L_t^\infty[0, \infty)} \leq \epsilon.$$

A few comments are in order. Our result constructs the co-dimension one center-stable manifold of initial data, for which the solutions of (1) close to the steady states stay close to the said steady states (and in fact converge at certain rate to zero). The results are important in several different regards - first, they show that the center-stable manifold is indeed a co-dimension one object, which is not *a priori* clear. Secondly, the actual construction, while of course implicit, relies on an implicit constraint (53), which we exhibit below and which is of independent interest. Thirdly, the paper develops new spectral and functional analytic tools for proving dispersive estimates for the perturbed Klein-Gordon evolution, which might prove useful in other related situations. Note that the explicit form of the decaying perturbative term will not be of particular importance for the argument below and in fact can be generalized to nonlinearities of the form  $N(|u|^2)u$ , with appropriate conditions on  $N$ .

## II. MAIN LINEAR ESTIMATES

The proof of the conditional stability theorem is based on a spectral decomposition or modulation argument and a contraction mapping argument in the appropriate spaces. The key is to define the spaces and the norm in such a way that one is not only able to close the argument, but also infer the decay rates. In this section, we will explain how to prove Lemma 2.1 and Lemma 2.2 which are the main tools needed to show the conditional stability result. The lemmas in this section will also help to understand the reason why we are choosing these particular spaces.

Let  $P_{a.c.}$  be a spectral projection associated to the continuous spectrum of  $\mathcal{H} = -\partial_x^2 + 1 - p\phi^{p-1}$ . It should be noted that for the results of this section, the particular form of the potential  $p\phi^{p-1}$  is unimportant and in fact they hold for any Schrödinger operator in the form  $\mathcal{L} = -\partial_{xx} + V$ , where  $V$  has sufficient polynomial decay at infinity and no eigenvalues or resonances at zero. For our particular case,  $V = -p\phi^{p-1}$ , these are satisfied by (3) and Lemma 1.2, Lemma 1.3.

*Lemma 2.1:* There exists a positive constant  $C$  such that for any  $g(t, x) \in \mathcal{S}(\mathbb{R}^2)$  and  $t \in \mathbb{R}$ ,

$$\left\| \int_0^t \frac{e^{-i(t-s)\sqrt{\mathcal{H}}}}{\sqrt{\mathcal{H}}} P_{a.c.} g(s, \cdot) ds \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \leq C \|g\|_{L_t^2 L_x^2(\mathbb{R}, \langle x \rangle^5 dx)} \quad (6)$$

*Lemma 2.2:* There exists a positive constant  $C$  such that for any  $g(t, x) \in \mathcal{S}(\mathbb{R}^2)$  and  $t \in \mathbb{R}$ ,

$$\left\| \langle x \rangle^{-3/2} \int_0^t e^{-i(t-s)\sqrt{\mathcal{H}}} P_{a.c.} g(s, \cdot) ds \right\|_{L_x^\infty L_t^2} \leq C \|\langle x \rangle^{3/2} g\|_{L_x^1 L_t^2} \quad (7)$$

*Remark 1:* In order to prove Lemma 2.1 and Lemma 2.2, we will prove Lemma 2.3 and Lemma 2.4 first.

*Lemma 2.3:* There exists a positive constant  $C$  such that for any  $f \in \mathcal{S}(\mathbb{R})$

$$\|\langle x \rangle^{-3/2} e^{-it\sqrt{\mathcal{H}}} P_{a.c.} f\|_{L_x^\infty L_t^2} \leq C \|f\|_{L^2} \quad (8)$$

where  $P_{a.c.}(\mathcal{H})$  is the spectral projection associated to the continuous spectrum of  $\mathcal{H} = -\partial_x^2 + 1 - p\phi^{p-1}$ .

*Lemma 2.4:* There exists a positive constant  $C$  such that for any  $g(t, x) \in \mathcal{S}(\mathbb{R}^2)$

$$\left\| \int_{\mathbb{R}} e^{is\sqrt{\mathcal{H}}} P_{a.c.} g(s, \cdot) ds \right\|_{L_x^2} \leq C \|\langle x \rangle^{3/2} g\|_{L_x^1 L_t^2} \quad (9)$$

In order to explain the difficulties involved and why we need to resort to the weighted estimates above, let us consider a very simple and naive model, which is nevertheless instructive. Consider a Schrödinger equation, which is not unlike our Klein-Gordon model

$$w_t + i\partial_{xx} w = w^2 \eta + w^p, \quad (t, x) \in \mathbb{R}^{1+1}.$$

with small data, where  $\eta$  is a rapidly decaying function and  $p \geq 5$ . It is not hard to check that the equation  $w_t + i\partial_{xx} w = w^p$ , one may apply the standard Strichartz estimates for  $e^{it\partial_{xx}}$  and be done with it very quickly. The addition of the highly-localized in  $x$  (but not rapidly decaying in time) term  $w^2 \eta$  presents a new challenge in one spatial dimension in particular. Note that for example in dimensions  $d \geq 3$ , this term can be handled by Strichartz estimates as well. This necessitates the introduction of the weighted estimates in Lemmas 2.1–2.4, which in essence permits exchanging this extra spatial decay for some extra time decay, just enough to close the fixed point arguments.

Before we embark on the proofs of these lemmas which are, as we saw, necessary ingredients in the proof of our main result, let us comment on the strategy and previous results in this direction. The Strichartz and decay estimates for Schrödinger operators with potentials have been subject of

intense investigations in the last twenty years, for example.<sup>5,7,11,18,29,30</sup> In these works, the authors prove decay/Strichartz estimates for the Schrödinger evolution  $e^{it(-\Delta + V)}$ , in different dimensions and under different assumptions on the potential  $V$  - like decay, absence of resonances/eigenvalues at zero energy. It has to be mentioned that the requirement for absence of zero eigenvalue and/or potential has been mostly removed in the one dimensional case (maybe at the expense of slightly worst decay requirements on  $V$ ).

As we mentioned above, in the concrete applications to asymptotic stability of one dimensional waves, the low powers become hard to handle and therefore, one needs the weighted estimates, similar to Lemma 2.3.

Although, we do not use any of these results in our work, we use heavily the approach of these earlier papers. We follow mostly the scheme of Mizumachi,<sup>18</sup> for the Schrödinger equation, which we consider a pioneering work in the area.

As is well-understood by now, one needs to split the estimates into high and low frequency regimes. In the high frequency regimes, one basically uses integration by parts (although this is accomplished by a non-trivial Born series expansion of the resolvents, together with a precise knowledge of the free resolvents). In low frequency, we have to heavily utilize known properties of the Jost solutions, which generate the perturbed resolvents directly. In all of this, we use what has become a standard way of approaching these weighted dispersive estimates. On the other hand, our arguments are being applied to study the Klein-Gordon's equation and as such, they are new and have subtleties, which are not present in the work of Mizumachi.

### III. PROOF OF MAIN TECHNICAL LEMMAS

#### A. Proof of Lemma 2.3 and Lemma 2.4

Define  $\varphi(x)$  to be a smooth function satisfying  $0 \leq \varphi(x) \leq 1$  for  $x \in \mathbf{R}$  and

$$\varphi(x) = \begin{cases} 1 & \text{if } x \geq 2 \\ 0 & \text{if } x \leq 1 \end{cases} \quad (10)$$

and let  $\varphi_M(x)$  be an even function satisfying  $\varphi_M(x) = \varphi(x - M)$  for  $x \geq 0$  and let  $\tilde{\varphi}_M(x) = 1 - \varphi_M(x)$ . Then define  $L := \mathcal{H} - 1 = -\partial_x^2 - p\phi^{p-1}$

$$P_{a.c.}e^{-it\sqrt{\mathcal{H}}}f = P_{a.c.}e^{-it\sqrt{L+1}}f = e^{-it\sqrt{L+1}}\varphi_M(\sqrt{L+1})f + P_{a.c.}e^{-it\sqrt{L+1}}\tilde{\varphi}_M(\sqrt{L+1})f \quad (11)$$

Let  $R(\lambda) = (\lambda - L)^{-1}$ , from Spectral Decomposition Theorem and Complex Analysis since

$$f(L) = \frac{1}{2\pi i} \int_{\wp} f(\lambda)(\lambda - L)^{-1} d\lambda \quad (12)$$

where  $\wp$  is the curve containing the absolute continuous spectrum of  $L$ , we have

$$\varphi_M(\sqrt{L+1})e^{-it\sqrt{L+1}}f = \frac{1}{2\pi i} \int_0^\infty e^{-it\sqrt{\lambda+1}}\varphi_M(\sqrt{\lambda+1})(R(\lambda - i0) - R(\lambda + i0))f d\lambda \quad (13)$$

and

$$P_{a.c.}e^{-it\sqrt{L+1}}\tilde{\varphi}_M(\sqrt{L+1})f = \frac{1}{2\pi i} \int_0^\infty e^{-it\sqrt{\lambda+1}}\tilde{\varphi}_M(\sqrt{\lambda+1})P_{a.c.}(R(\lambda - i0) - R(\lambda + i0))f d\lambda \quad (14)$$

By change of variables  $\mu := \sqrt{\lambda+1}$ , (13) becomes

$$= \frac{1}{\pi i} \int_{-\infty}^\infty \chi_{[1,\infty]}e^{-it\mu}\varphi_M(\mu)(R(\mu^2 - 1 - i0) - R(\mu^2 - 1 + i0))\mu f d\mu \quad (15)$$

Applying integration by parts for  $j$  times, we get

$$\begin{aligned} & \varphi_M(\sqrt{L+1})e^{-it\sqrt{L+1}}f \\ &= \frac{(it)^{-j}}{\pi i} \int_{-\infty}^{\infty} e^{-it\mu} \partial_{\mu}^j \{ \chi_{[1,\infty)} \varphi_M(\mu) (R(\mu^2 - 1 - i0) - R(\mu^2 - 1 + i0)) \mu \} f d\mu \end{aligned} \quad (16)$$

in  $S'_x(\mathbf{R})$  for any  $t \neq 0$  and  $f \in S_x(\mathbf{R}^2)$ . Since

$$\|\partial_{\lambda}^j P_{a.c.} R(\lambda \pm i0)\|_{B(L^{2,(j+1)/(2+0)}, L^{2,-(j+1)/(2-0)})} \lesssim \langle \lambda \rangle^{-(j+1)/2} \quad (17)$$

the integral is absolutely convergent in  $L_x^{2,-(j+1)/2}$  for  $j \geq 2$ .

Suppose  $g(t, x) = g_1(t)g_2(x)$  where  $g_1 \in C_0^{\infty}(\mathbf{R} - \{0\})$ ,  $g_2 \in S(\mathbf{R})$ . Define

$$\langle u_1, u_2 \rangle_x := \int_{-\infty}^{\infty} u_1(x)u_2(x)dx \quad (18)$$

$$\langle v_1, v_2 \rangle_{t,x} := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_1(t, x)v_2(t, x)dxdt \quad (19)$$

Thus

$$\langle \varphi_M(\sqrt{L+1})e^{-it\sqrt{L+1}}f, g \rangle_{t,x} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_M(\sqrt{L+1})e^{-it\sqrt{L+1}}f g_1(t)g_2(x)dxdt \quad (20)$$

Using (16), we get

$$\begin{aligned} & \langle \varphi_M(\sqrt{L+1})e^{-it\sqrt{L+1}}f, g \rangle_{t,x} \\ &= \frac{1}{\pi i} \int_{-\infty}^{\infty} dt (it)^{-j} g_1(t) \int_{-\infty}^{\infty} d\mu e^{-it\mu} \partial_{\mu}^j \{ \chi_{[1,\infty)} (R(\mu^2 - 1 - i0) - R(\mu^2 - 1 + i0)) \varphi_M(\mu) \mu f, g_2 \}_x \end{aligned}$$

By Fubini's Theorem

$$= \frac{1}{\pi i} \int_{-\infty}^{\infty} d\mu \partial_{\mu}^j \{ \chi_{[1,\infty)} (R(\mu^2 - 1 - i0) - R(\mu^2 - 1 + i0)) \varphi_M(\mu) \mu f, g_2 \}_x \int_{-\infty}^{\infty} dt (it)^{-j} e^{-it\mu} g_1(t)$$

Doing integration by parts for  $j$  times

$$= \frac{\sqrt{2}}{\sqrt{\pi i}} \int_{-\infty}^{\infty} d\mu (\mathcal{F}_t g_1)(\mu) \langle \chi_{[1,\infty)} (R(\mu^2 - 1 - i0) - R(\mu^2 - 1 + i0)) \varphi_M(\mu) \mu f, g \rangle_x$$

From Fubini's Theorem

$$= \frac{\sqrt{2}}{\sqrt{\pi i}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\mu \langle \chi_{[1,\infty)} (R(\mu^2 - 1 - i0) - R(\mu^2 - 1 + i0)) \varphi_M(\mu) \mu f \mathcal{F}_t g(\mu, x) \rangle$$

Using Plancherel's Theorem and Cauchy Schwartz Inequality

$$\begin{aligned} & |\langle \varphi_M(\sqrt{L+1})e^{-it\sqrt{L+1}}f, g \rangle_{t,x}| \\ & \leq \frac{\sqrt{2}}{\sqrt{\pi i}} \|\varphi_M(\mu) \mu (R(\mu^2 - 1 - i0) - R(\mu^2 - 1 + i0)) f\|_{L_x^{\infty} L_{\mu}^2} \|g(\mu, x)\|_{L_x^1 L_{\mu}^2} \end{aligned}$$

Similarly

$$\begin{aligned} & |\langle P_{a.c.} e^{-it\sqrt{L+1}} \tilde{\varphi}_M(\sqrt{L+1}) f, g \rangle_{t,x}| \\ & \leq \frac{\sqrt{2}}{\sqrt{\pi i}} \|\langle x \rangle^{-3/2} \tilde{\varphi}_M(\mu) P_{a.c.} (R(\mu^2 - 1 - i0) - R(\mu^2 - 1 + i0)) \mu f\|_{L_x^{\infty} L_{\mu}^2} \|\langle x \rangle^{3/2} g(\mu, x)\|_{L_x^1 L_{\mu}^2} \end{aligned}$$

If we combine these two and assuming the next two inequalities (21) and (22) hold

$$\|\varphi_M(\mu)(R(\mu^2 - 1 - i0) - R(\mu^2 - 1 + i0))\mu f\|_{L_x^\infty L_\mu^2} \leq C\|f\|_{L^2} \quad (21)$$

$$\|\langle x \rangle^{-3/2} \tilde{\varphi}_M(\mu) P_{a.c.}(R(\mu^2 - 1 - i0) - R(\mu^2 - 1 + i0))\mu f\|_{L_x^\infty L_\mu^2} \leq C\|f\|_{L^2} \quad (22)$$

we get

$$|\langle x \rangle^{-3/2} \langle e^{-it\sqrt{L+1}} P_{a.c.} f, g \rangle_{t,x}| \leq C\|f\|_{L^2} \|g\|_{L_x^1 L_\mu^2} \quad (23)$$

Since  $\mathcal{C}_0^\infty(\mathbb{R}_t - \{0\}) \otimes \mathcal{S}(\mathbb{R}_x)$  is dense in  $L_x^1 L_t^2$  and by duality principle

$$\|\langle x \rangle^{-3/2} e^{-it\sqrt{L+1}} P_{a.c.} f\|_{L_x^\infty L_\mu^2} \leq C\|f\|_{L^2} \quad (24)$$

Now we will prove (21) then (22) in order to complete the proof of the lemma. We will use Green's functions to show (21), Scattering Theory and Jost functions to prove (22).

## B. Proof of (21): High energy estimate

Let  $R_0(\lambda) = (\lambda + \partial_x^2)^{-1}$  and  $G_1(x, k) = \frac{e^{ik|x|}}{2ik}$ , and  $\lambda = k^2$  with  $k \geq 0$  and  $V := -p\phi^p - 1$ . Then  $R_0(\lambda \pm i0)\delta = G_1(x, \mp k)$ . If  $M$  is sufficiently large, we have

$$R(\lambda \pm i0) = \sum_{j=0}^{\infty} R_0(\lambda \pm i0)(V R_0(\lambda \pm i0))^j u \quad (25)$$

for  $\lambda \in \mathbb{R}$  with  $|\lambda| > M$  and  $u \in \mathcal{S}(\mathbb{R})$  since

$$\|\langle x \rangle^{-1} R_0(\lambda \pm i0) \langle x \rangle^{-1}\|_{B(L^2(\mathbb{R}))} \lesssim \langle \lambda \rangle^{-1/2} \quad (26)$$

The sum is absolutely convergent because

$$\begin{aligned} R(\lambda \pm i0)u &= R_0(\lambda \pm i0)u + R_0(\lambda \pm i0)V R_0(\lambda \pm i0)u + \dots \\ &= \langle x \rangle \langle x \rangle^{-1} R_0(\lambda \pm i0) \langle x \rangle^{-1} \langle x \rangle u \\ &\quad + \langle x \rangle \langle x \rangle^{-1} R_0(\lambda \pm i0) \langle x \rangle^{-1} \langle x \rangle V \langle x \rangle \langle x \rangle^{-1} R_0(\lambda \pm i0) \langle x \rangle^{-1} \langle x \rangle u + \dots \end{aligned}$$

Since  $V$  is exponentially decreasing and  $u \in \mathcal{S}(\mathbb{R})$ , the absolute sum in  $L^2$  is bounded by

$C \sum_{j=1}^{j=\infty} \langle \lambda \rangle^{-j/2}$ . Since  $|\lambda| > M$  and  $M$  is large enough, the geometric series converges. Now if we assign  $\lambda = \mu^2 - 1$ , then we can write

$$\begin{aligned} \|\varphi_M(\mu)R(\mu^2 - 1 \pm i0)\mu u\|_{L_x^\infty L_\mu^2} &\leq \|\varphi_M(\mu)R_0(\mu^2 - 1 \pm i0)\mu u\|_{L_x^\infty L_\mu^2} \\ &\quad + \sum_{n=1}^{\infty} \|\varphi_M(\mu)F_{1,n}(x, \mp k)\|_{L_x^\infty L_\mu^2} \end{aligned}$$

where

$$F_{1,n}(x, \mp k) := R_0(\mu^2 - 1 \pm i0)(V R_0(\mu^2 - 1 \pm i0))^n \mu u(x) \quad (27)$$

$$\begin{aligned} \|\varphi_M(\mu)R_0(\mu^2 - 1 \pm i0)\mu u\|_{L_x^\infty L_\mu^2}^2 &= \sup_x \int_{\mathbf{R}} |\varphi_M(\mu)R_0(\mu^2 - 1 \pm i0)\mu u|^2 d\mu \\ &= \sup_x \int_{\mathbf{R}} \frac{k}{\sqrt{k^2 + 1}} |\varphi_M(\sqrt{k^2 + 1})R_0(k^2 \pm i0)\sqrt{k^2 + 1}u|^2 dk \\ &= \sup_x \int_{\mathbf{R}} k\sqrt{k^2 + 1} |\varphi_M(\sqrt{k^2 + 1})(G_1(\cdot, \mp k) * u)(x)|^2 dk \\ &\lesssim \sup_x \int_{\mathbf{R}} \left( \left| \int_x^\infty u(y)e^{\pm iky} dy \right|^2 + \left| \int_{-\infty}^x u(y)e^{\mp iky} dy \right|^2 \right) dk \\ &\lesssim \|u\|_{L_x^2}^2 \end{aligned}$$

Similarly one can write

$$F_{1,n}(x, \pm k) = \int_{\mathbf{R}^{n+1}} G_1(x - x_1, \pm k) \prod_{j=1}^n (V(x_j)G_1(x_j - x_{j+1}, \pm k))\sqrt{k^2 + 1}u(x_{n+1})dx_1 \dots dx_{n+1} \quad (28)$$

Since

$$\int_{\mathbf{R}} G_1(x_n - x_{n+1})u(x_{n+1})dx_{n+1} = G_1(x_n) * u(x_n) \quad (29)$$

by Minkowski's Inequality, we get

$$\begin{aligned} \|\varphi_M(\mu)F_{1,n}(x, \pm k)\|_{L_\mu^2} &= \left( \int_{\mathbf{R}} |\varphi_M(\mu)F_{1,n}(x, \pm k)|^2 d\mu \right)^{1/2} \lesssim \int_{\mathbf{R}^n} \prod_{j=1}^n V(x_j)dx_1 \dots dx_n \\ &\times \left( \int_{\mathbf{R}} k(k^2 + 1)|\varphi_M(\sqrt{k^2 + 1})G_1(x - x_1) \dots G_1(x - x_n)|^2 |(G_1 * u)(x_n)|^2 dk \right)^{1/2} \\ &\lesssim \|V\|_{L^1}^n \sup_{x_n} \left( \int_{\mathbf{R}} k^{-2n} k(k^2 + 1)|\varphi_M(\sqrt{k^2 + 1})(G_1 * u)(x_n)|^2 dk \right)^{1/2} \\ &\lesssim \|V\|_{L^1}^n M^{(-2n+1)/2} \|u\|_{L^2} \quad \text{for } n \geq 1 \end{aligned}$$

Since  $V \in L^1(\mathbb{R})$ ,  $u \in \mathcal{S}(\mathbb{R})$  and  $M$  is sufficiently large, we have

$$\begin{aligned} \|\varphi_M(\mu)R_0(\mu^2 - 1 \mp i0)\mu u\|_{L_x^\infty L_\mu^2} &\lesssim \|u\|_{L^2} + \sum_{n=1}^{\infty} \|V\|_{L^1}^n M^{-n+1/2} \|u\|_{L^2} \\ &\lesssim \|u\|_{L^2} \end{aligned}$$

### C. Proof of (22): Low energy estimate

This section is based on Jost functions and Scattering Theory. Let  $f_1(x, k)$  and  $f_2(x, k)$  be the solutions to  $Lu = k^2 u$  satisfying

$$\lim_{x \rightarrow \infty} |e^{-ikx} f_1(x, k) - 1| = 0, \quad \lim_{x \rightarrow -\infty} |e^{ikx} f_2(x, k) - 1| = 0 \quad (30)$$

Define

$$m_1(x, k) := e^{-ikx} f_1(x, k), \quad m_2(x, k) := e^{ikx} f_2(x, k)$$



Then

$$m_1(x, k) = 1 + \int_x^\infty \frac{e^{2ik(y-x)}}{2ik} V(y) m_1(y, k) dy$$

$$m_2(x, k) = 1 + \int_{-\infty}^x \frac{e^{2ik(x-y)}}{2ik} V(y) m_2(y, k) dy$$

<sup>13</sup> tells that for  $x \in \mathbb{R}$  and  $k \in \mathbb{C}$  with nonnegative imaginary part,

$$|m_1(x, k) - 1| \lesssim \langle k \rangle^{-1} (1 + \max(-x, 0)) \int_x^\infty \langle y \rangle |V(y)| dy \quad (31)$$

$$|m_2(x, k) - 1| \lesssim \langle k \rangle^{-1} (1 + \max(x, 0)) \int_{-\infty}^x \langle y \rangle |V(y)| dy \quad (32)$$

For every  $\delta > 0$ , there exists  $C_\delta > 0$  such that for every  $x \in \mathbb{R}$  and  $k \in \mathbb{C}$  with nonnegative imaginary part and  $|k| \geq \delta$

$$|m_1(x, k) - 1| \leq C_\delta \int_x^\infty |V(y)| dy \quad (33)$$

$$|m_2(x, k) - 1| \leq C_\delta \int_{-\infty}^x |V(y)| dy \quad (34)$$

The resolvent operator  $R(\lambda \pm i0)$  with  $\lambda = k^2$  has the kernel

$$K_\pm(x, y, k) = \begin{cases} -\frac{f_1(x, \pm k)f_2(y, \pm k)}{W(\pm k)} & \text{if } x > y \\ -\frac{f_2(x, \pm k)f_1(y, \pm k)}{W(\pm k)} & \text{if } x < y \end{cases} \quad (35)$$

where  $W(k) = f_1'(x, k)f_2(x, k) - f_1(x, k)f_2'(x, k)$  where the Wronskian  $W(k)$  is independent of  $x$ .

$$R(\lambda \pm i0)u = -\frac{f_1(x, \pm k)}{W(\pm k)} (I_1 + I_2 + I_3) - \frac{f_2(x, \pm k)}{W(\pm k)} (II_1 + II_2)$$

where  $I_1(k) = \int_{-\infty}^0 e^{-iky} u(y) dy$ ,  $I_2(k) = \int_0^\infty e^{-iky} (m_2(y, k) - 1) u(y) dy$ ,  $I_3(k) = \int_0^x f_2(y, k) u(y) dy$  and  $II_1(k) = \int_x^\infty e^{iky} u(y) dy$ ,  $II_2(k) = \int_x^\infty e^{iky} (m_2(y, k) - 1) u(y) dy$ .

**Bound for  $I_1$ :** Assuming  $x > 0$ , (31) and (32) imply that

$$\sup_{x>0} (|f_1(x, k)| + \langle x \rangle^{-1} |f_2(x, k)|) < \infty \quad (36)$$

Then

$$|I_1| = \left| \int_0^x f_2(y, k) u(y) dy \right| \lesssim \int_0^x \langle y \rangle |u(y)| dy \lesssim \left( \int_0^x \langle y \rangle^2 dy \right)^{1/2} \left( \int_0^x |u(y)|^2 dy \right)^{1/2}$$

$$\lesssim \langle x \rangle^{3/2} \|u\|_{L^2}$$

By using (31) and (32), Cauchy-Schwartz Inequality and the properties of Fourier Transform, one can also bound  $I_2, I_3, II_1$  and  $II_2$  by  $C\|u\|_{L^2}$ . Then since  $W(k) \neq 0$  for every  $k \in \mathbb{R}$  and  $\tilde{\varphi}_M(k)$  is compactly supported, it follows that

$$\|\tilde{\varphi}_M(\mu) P_{a.c.} R(\mu^2 - 1 \pm i0) \mu f\|_{L_x^\infty L_\mu^2}$$

$$= \sup_x \left( \int_{\mathbb{R}} |k| (k^2 + 1) |\tilde{\varphi}_M(k) \int_{\mathbb{R}} K_\pm(x, y, k) u(y) dy|^2 dk \right)^{1/2}$$

$$\lesssim \langle x \rangle^{3/2} \|u\|_{L^2}$$

This finishes the proof of Lemma 2.3.

In fact, the proof of Lemma 2.4 relies on a simple duality argument, based on Lemma 2.3. Define  $Tf := \langle x \rangle^{-3/2} e^{-it\sqrt{\mathcal{H}}} P_{a.c.} f$ . From Lemma 2.3, we have  $\|Tf\|_{L_x^\infty L_t^2} \leq C \|f\|_{L^2}$ . Then using Fubini's Theorem and Duality Principle we get

$$|\int_{\mathbb{R}} \int_{\mathbb{R}} \langle x \rangle^{-3/2} e^{-it\sqrt{\mathcal{H}}} P_{a.c.} f h dx dt| = |\langle f, \int_{\mathbb{R}} dt e^{it\sqrt{\mathcal{H}}} P_{a.c.} \langle x \rangle^{-3/2} h \rangle_x| \leq C \|f\|_{L^2} \|h\|_{L_x^1 L_t^2} \quad (37)$$

If we define  $g := \langle x \rangle^{-3/2} h$ , then (9) follows by duality principle.

#### D. Proof of Lemma 2.1 and Lemma 2.2

First, we need the following

*Definition 3.1:* We say that a pair  $(q, r)$  is KG admissible (sharp KG admissible respectively), if  $q, r \geq 2$ :  $2/q + d/r \leq d/2$  ( $q, r \geq 2$ :  $2/q + d/r = d/2$  respectively) and  $(q, r, d) \neq (2, \infty, 2)$ .

*Lemma 3.2:* (Lemma 2.1 in Ref. 19 with  $\sigma = d, \lambda = (d + 2)/2$ ). Let  $(q, r), (q_1, r_1)$  be both KG admissible pairs and  $s \geq 0$ . Then, for  $\mathcal{H}_0 = -\Delta + 1$ ,

$$\begin{aligned} \|e^{it\sqrt{\mathcal{H}_0}} f\|_{L_t^q W_x^{r,s}} &\leq C \|f\|_{H^{s+\frac{d+2}{2}(\frac{1}{r}-\frac{1}{q})}} \\ \left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}_0})}{\sqrt{\mathcal{H}_0}} G(s, \cdot) ds \right\|_{L_t^q W_x^{r,s}} &\leq C \|G\|_{L_t^{q_1} W_x^{r_1, s-1+\frac{d+2}{2}(\frac{1}{r_1}-\frac{1}{q})}} \end{aligned}$$

By using wave operators, as in Ref. 26, one can show the same inequalities for  $\mathcal{H} = -\partial_x^2 + 1 - p\phi^{p-1}$ .

*Proof of Lemma 2.1:* From Strichartz estimates for the Klein Gordon equation, we have

$$\|e^{-it\sqrt{\mathcal{H}}} P_{a.c.} f\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \leq C \|f\|_{H^1} \quad (38)$$

Similarly, we get

$$\left\| \frac{e^{-it\sqrt{\mathcal{H}}}}{\sqrt{\mathcal{H}}} P_{a.c.} f \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \leq C \|f\|_{L^2} \quad (39)$$

and from Lemma 2.4, we know that

$$\left\| \int_{\mathbb{R}} e^{is\sqrt{\mathcal{H}}} P_{a.c.} g(s, \cdot) ds \right\|_{L_x^2} \leq C \|\langle x \rangle^{3/2} g\|_{L_x^1 L_t^2} \quad (40)$$

Let

$$Tg(t) = \int_{\mathbb{R}} \frac{e^{-i(t-s)\sqrt{\mathcal{H}}}}{\sqrt{\mathcal{H}}} P_{a.c.} g(s) ds \quad (41)$$

Choose

$$f := \int_{\mathbb{R}} e^{is\sqrt{\mathcal{H}}} P_{a.c.} g(s) ds \in L^2(\mathbb{R}) \quad (42)$$

Then using (39) and (40) and Cauchy-Schwartz inequality, we get

$$\begin{aligned} \|Tg\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} &= \left\| \int_{\mathbb{R}} \frac{e^{-i(t-s)\sqrt{\mathcal{H}}}}{\sqrt{\mathcal{H}}} P_{a.c.} g(s, \cdot) ds \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \\ &\leq C \|f\|_{L^2} \lesssim C \|\langle x \rangle^{3/2} g\|_{L_x^1 L_t^2} \leq \|\langle x \rangle^{5/2} g\|_{L_x^2 L_t^2} \|\langle x \rangle^{-1}\|_{L_x^2} \\ &\leq C \|g\|_{L_t^2 L_x^2(\mathbb{R}; \langle x \rangle^5 dx)} \end{aligned}$$

Using the result in Ref. 12, it follows that

$$\left\| \int_{s < t} e^{-i(t-s)\sqrt{\mathcal{H}}} P_{a.c.} g(s) ds \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \lesssim \|g\|_{L_t^2 L_x^2(\mathbb{R}; \langle x \rangle^5 dx)} \quad (43)$$

Thus we complete the proof of Lemma 2.1.  $\square$

*Proof of Lemma 2.2:* In order to show Lemma 2.2, we shall need two modifications of results appearing in Ref. 18. These will be needed to control various terms, arising in the analysis of the estimate (7).

The first result is stated in Ref. 18 for self-adjoint operators  $H = -\partial_x^2 + V$ , but in fact, it is applicable for any self-adjoint operator acting on  $L^2$ .

*Proposition 3.3: (Lemma 11, Ref. 18)* Let  $H$  be a self-adjoint operator and  $g(t, x) = g_1(t)g_2(x)$ . Define the function

$$U(t, x) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\lambda} \check{g}_1(\lambda) \{R(\lambda - i0) + R(\lambda + i0)\} [P_{a.c.}(H)g_2] d\lambda,$$

where  $\check{g}_1$  is the inverse Fourier transform of  $g_1$ . Then

$$\begin{aligned} U(t, x) &= 2 \int_0^t e^{-i(t-s)H} P_{a.c.}(H)g(s, \cdot) ds + \int_{-\infty}^0 e^{-i(t-s)H} P_{a.c.}(H)g(s, \cdot) ds \\ &\quad - \int_0^\infty e^{-i(t-s)H} P_{a.c.}(H)g(s, \cdot) ds \end{aligned}$$

One can obtain similar results for expressions in the form  $\int_0^t e^{-i(t-s)\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H})g(s, \cdot) ds$  with  $g(t, x) = g_1(t)g_2(x)$ . Namely, based on the argument in Proposition 3.3, we have

$$\begin{aligned} &\frac{i}{2\sqrt{2\pi}} \int_1^\infty e^{-it\sqrt{\lambda}} \check{g}_1(\lambda) \{R(\lambda - i0) + R(\lambda + i0)\} [P_{a.c.}(\mathcal{H})g_2] d\lambda = \\ &\int_0^t e^{-i(t-s)\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H})g(s, \cdot) ds - \int_0^\infty e^{-i(t-s)\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H})g(s, \cdot) ds + \int_{-\infty}^0 e^{-i(t-s)\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H})g(s, \cdot) ds \end{aligned}$$

Solving for the Duhamel's operator, associated with our evolution, we get the following formula

$$\begin{aligned} &\int_0^t e^{-i(t-s)\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H})g(s, \cdot) ds = \\ &= \frac{i}{2\sqrt{2\pi}} \int_1^\infty e^{-it\sqrt{\lambda}} \check{g}_1(\lambda) \{R(\lambda - i0) + R(\lambda + i0)\} [P_{a.c.}(\mathcal{H})g_2] d\lambda + \\ &+ \frac{1}{2} \int_0^\infty e^{-i(t-s)\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H})g(s, \cdot) ds - \frac{1}{2} \int_{-\infty}^0 e^{-i(t-s)\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H})g(s, \cdot) ds \end{aligned}$$

Combining Lemma 8 and Lemma 10 from Ref. 18 yields the following. Note that there is a missing  $P_{a.c.}(H)$  from the statement of both lemmas in Ref. 18.

*Proposition 3.4:* Let  $H = -\partial_x^2 + V(x)$ , where  $V(x)$  is a real valued potential, which decays sufficiently fast. Then

$$\sup_\lambda \| \langle x \rangle^{-1} R_H(\lambda \pm i0) P_{a.c.}(H) u \|_{L_x^\infty} \leq \frac{C}{\langle \lambda \rangle^{1/2}} \| \langle x \rangle u \|_{L_x^1}. \quad (44)$$

**Note:** The constant  $\langle \lambda \rangle^{-1/2}$  in (44) is not stated in Lemma 8,<sup>18</sup> (which is the high-frequency version regime, i.e.,  $\lambda \gg 1$ ), but it is very explicit in the estimates there.

We are now ready to proceed with the proof of Lemma 2.2. First, it is standard that in order to establish (7), it suffices to consider functions  $g(t, x) = g_1(t)g_2(x)$ . Therefore, in view of our formula

for  $\int_0^t e^{-i(t-s)\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H})g(s, \cdot)ds$ , it remains to establish

$$\|\langle x \rangle^{-1} \int_1^\infty e^{-it\sqrt{\lambda}} \check{g}_1(\lambda) R(\lambda \pm i0) [P_{a.c.}(\mathcal{H})g_2] d\lambda\|_{L_x^\infty L_t^2} \leq C \|g_1\|_{L_t^2} \|\langle x \rangle g_2\|_{L_x^1} \quad (45)$$

$$\|\langle x \rangle^{-3/2} \int_0^\infty e^{-i(t-s)\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H})g(s, \cdot)ds\|_{L_x^\infty L_t^2} \leq C \|\langle x \rangle^{3/2} g\|_{L_x^1 L_t^2} \quad (46)$$

$$\|\langle x \rangle^{-3/2} \int_{-\infty}^0 e^{-i(t-s)\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H})g(s, \cdot)ds\|_{L_x^\infty L_t^2} \leq C \|\langle x \rangle^{3/2} g\|_{L_x^1 L_t^2} \quad (47)$$

The proofs of (46) and (47) are similar, so we concentrate on (46). We have from (8) and (9)

$$\begin{aligned} & \|\langle x \rangle^{-3/2} \int_0^\infty e^{-i(t-s)\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H})g(s, \cdot)ds\|_{L_x^\infty L_t^2} = \\ &= \|\langle x \rangle^{-3/2} e^{-it\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H}) \int_0^\infty e^{is\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H})g(s, \cdot)ds\|_{L_x^\infty L_t^2} \leq \\ &\leq C \|\int_0^\infty e^{is\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H})g(s, \cdot)ds\|_{L_x^2} \leq C \|\langle x \rangle^{3/2} g\|_{L_x^1 L_t^2}. \end{aligned}$$

Regarding (45), we have by Plancherel's theorem in the time variable and Cauchy-Schwartz inequality that

$$\begin{aligned} & \|\langle x \rangle^{-1} \int_1^\infty e^{-it\sqrt{\lambda}} \check{g}_1(\lambda) R(\lambda \pm i0) [P_{a.c.}(\mathcal{H})g_2] d\lambda\|_{L_x^\infty L_t^2} \leq \\ &\leq 2 \sup_x |\langle x \rangle^{-1}| \|\int_1^\infty e^{-it\mu} \mu \check{g}_1(\mu^2) R(\mu^2 \pm i0) [P_{a.c.}(\mathcal{H})g_2] d\mu\|_{L_t^2} \leq \\ &\leq C (\int_{-\infty}^\infty |\mu| |\check{g}_1(\mu^2)|^2 d\mu)^{1/2} \sup_\mu |\mu|^{1/2} \sup_x |\langle x \rangle^{-1} R(\mu^2 \pm i0) (P_{a.c.}(\mathcal{H})g_2)(x)| \end{aligned}$$

From (44), we have  $\|\langle x \rangle^{-1} R(\mu^2 \pm i0) P_{a.c.}(\mathcal{H})\langle x \rangle^{-1}\|_{L_x^1 \rightarrow L_x^\infty} \leq C < \mu >^{-1}$ , whence

$$\sup_x |\langle x \rangle^{-1} R(\mu^2 \pm i0) (P_{a.c.}(\mathcal{H})g_2)(x)| \leq C \langle \mu \rangle^{-1} \| < x > g_2 \|_{L_x^1}.$$

Overall, observing that  $(\int_{-\infty}^\infty |\mu| |\check{g}_1(\mu^2)|^2 d\mu)^{1/2} \leq \|\check{g}_1\|_{L^2} = \|g_1\|_{L_t^2}$  and  $|\mu|^{1/2} \langle \mu \rangle^{-1} < 1$ , we conclude

$$\|\langle x \rangle^{-1} \int_1^\infty e^{-it\sqrt{\lambda}} \check{g}_1(\lambda) R(\lambda \pm i0) [P_{a.c.}(\mathcal{H})g_2] d\lambda\|_{L_x^\infty L_t^2} \leq C \|g_1\|_{L_t^2} \|\langle x \rangle g_2\|_{L_x^1},$$

which is (45).  $\square$

#### IV. PROOF OF THE THEOREM 1.4

In this section, we will prove the conditional stability result by applying the fixed point theorem. We will set the contraction map and the function spaces. In order to prove the contraction mapping theorem, for the decay estimates, we will apply Lemma 2.1 and Lemma 2.2 and for the Strichartz estimates, we will use Lemma 3.2.

##### A. Analysis of $a(t)$ and $z(t)$ equations

Taking the ansatz (5) into (2), we get

$$z_{tt} + \mathcal{H}z + \psi(a''(t) - \sigma^2 a(t)) - F(t, x) = 0 \quad (48)$$

where

$$F(t, x) = |\phi + a(t)\psi + \mathbf{z}|^{p-1}(\phi + a(t)\psi + \mathbf{z}) - \phi^p - p\phi^{p-1}(a(t)\psi + \mathbf{z}(t)) \quad (49)$$

Taking the spectral projections, we derive the equations

$$a''(t) - \sigma^2 a(t) - \langle F(t, \cdot), \psi \rangle = 0 \quad (50)$$

$$\mathbf{z}_{tt} + \mathcal{H}\mathbf{z} - P_{a.c.}[F] = 0 \quad (51)$$

The explicit solution of (50) is in the form

$$a(t) = \cosh(\sigma t)a(0) + \frac{1}{\sigma} \sinh(\sigma t)a'(0) + \frac{1}{\sigma} \int_0^t \sinh(\sigma(t-s)) \langle F(s, \cdot), \psi \rangle ds \quad (52)$$

Note that, if we separate the exponentially growing terms from the exponentially decaying ones, we come up with

$$a(t) = \frac{e^{\sigma t}}{2} \left[ a(0) + \frac{1}{\sigma} a'(0) + \frac{1}{\sigma} \int_0^t e^{-\sigma s} \langle F(s, \cdot), \psi \rangle ds \right] + \text{exponentially decaying term.}$$

In order to have a vanishing solution, we must have  $a(t) \rightarrow 0$ , and so, at the very least, we must ensure (by taking appropriate initial data)

$$a(0) + \frac{1}{\sigma} a'(0) + \frac{1}{\sigma} \int_0^\infty e^{-\sigma s} \langle F(s, \cdot), \psi \rangle ds = 0. \quad (53)$$

The non-explicit non-linear equation (53) defines the center stable manifold as we shall show below and in that sense, it is useful in its own right. It also shows (modulo the successful completion of our argument) that it is co-dimension one. This, although being heuristically expected (due to the presence of a single unstable direction of the linearized operator), is not at all an obvious statement.

According to our definitions  $a(0) = \langle f_1 + h\psi, \psi \rangle = h + \langle f_1, \psi \rangle$ . Similarly,  $a'(0) = \langle f_2, \psi \rangle$ . Taking into account  $\langle f_1 + \frac{1}{\sigma} f_2, \psi \rangle = 0$ , we have no choice, but to set (as in Ref. 26)

$$h(f_1, f_2) = -\frac{1}{\sigma} \int_0^\infty e^{-\sigma s} \langle F(m(s)), \psi \rangle ds \quad (54)$$

Thus, (52) becomes equivalent to

$$a(t) = \frac{e^{-t\sigma}}{2} [a(0) - \frac{1}{\sigma} a'(0)] - \frac{1}{2\sigma} \int_0^t e^{-\sigma(t-s)} \langle F(s, \cdot), \psi \rangle ds - \frac{1}{2\sigma} \int_t^\infty e^{\sigma(t-s)} \langle F(s, \cdot), \psi \rangle ds \quad (55)$$

Taking into account  $P_{a.c.}(\mathcal{H})\psi = 0$ , the explicit solution of (51) is in the form

$$\mathbf{z}(t) = \cos(t\sqrt{\mathcal{H}})P_{a.c.}f_1 + \frac{\sin(t\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}}P_{a.c.}f_2 + \int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}}P_{a.c.}[F(s, \cdot)]ds \quad (56)$$

## B. Setting the contraction map and the function spaces

Let  $\Lambda$  be a contraction map defined as  $\Lambda: X \rightarrow X$  such that  $\Lambda(m) = \tilde{m}$  where  $m := (h, a(t), z(t))$  defined as (54)–(56) and  $\tilde{m} = (\tilde{h}, \tilde{a}(t), \tilde{z}(t))$

$$\tilde{h} := -\frac{1}{\sigma} \int_0^\infty e^{-\sigma s} \langle F(m(s)), \psi \rangle ds,$$

$$\tilde{a}(t) := \frac{e^{-t\sigma}}{2} [a(0) - \frac{1}{\sigma} a'(0)] - \frac{1}{2\sigma} \int_0^t e^{-\sigma(t-s)} \langle F(s, \cdot), \psi \rangle ds - \frac{1}{2\sigma} \int_t^\infty e^{\sigma(t-s)} \langle F(s, \cdot), \psi \rangle ds,$$

$$\tilde{\mathbf{z}}(t) := \cos(t\sqrt{\mathcal{H}})P_{a.c.}f_1 + \frac{\sin(t\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}}P_{a.c.}f_2 + \int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}}P_{a.c.}[F(s, \cdot)]ds.$$

Let the norm on  $X$  be defined as  $\|m\|_X := \max(M_0(m), M_1(m), M_2(m))$  such that

$$M_0(m) := |h|$$

$$M_1(m) := \|a\|_{L_t^3([0,\infty)) \cap L_t^\infty([0,\infty))}$$

$$M_2(m) = \|z\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1 \cap L_x^\infty(\mathbb{R}; \langle x \rangle^{-3/2} dx) L_t^2}$$

Our goal is to show that  $\Lambda$  is a contraction map defined on the Banach Space  $X$ , whose fixed point will be the desired solution.

### C. Estimating $M_0(\tilde{m})$

$$M_0(\tilde{m}) = |\tilde{h}| \leq \frac{1}{\sigma} \int_0^\infty e^{-\sigma s} |\langle F(m(s)), \psi \rangle| ds \quad (57)$$

From Proposition 3 in Ref. 26, we have

$$|F(t, x)| \leq C_p (\phi^{p-2} (|a(t)|^2 \psi^2 + |z(t)|^2) + |a(t)|^p \psi^p + |z(t)|^p) \quad (58)$$

Then it follows

$$|\langle F(m(s)), \psi \rangle| \leq C (|a(s)|^2 + \|z(s, \cdot)\|_{L_x^2}^2 + |a(t)|^p + \|z(s, \cdot)\|_{L_x^2}^p) \quad (59)$$

where  $C$  depends on various  $L^w$  norms of the decaying functions  $\phi, \psi$ . It follows that

$$\begin{aligned} M_0(\tilde{m}) &\leq \frac{C}{\sigma} \int_0^\infty e^{-\sigma s} (|a(s)|^2 + \|z(s, \cdot)\|_{L_x^2}^2 + |a(t)|^p + \|z(s, \cdot)\|_{L_x^2}^p) ds \\ &\leq \frac{C}{\sigma^2} (\|a\|_{L^\infty}^2 + \|z\|_{L_t^\infty L_x^2}^2 + \|a\|_{L^\infty}^p + \|z\|_{L_t^\infty L_x^2}^p) \\ &\leq \frac{C}{\sigma^2} (M_1(m)^2 + M_2(m)^2 + M_1(m)^p + M_2(m)^p) \leq \frac{2C}{\sigma^2} (\epsilon^2 + \epsilon^p) \leq \min(1, \sigma) \frac{\epsilon}{10} \end{aligned}$$

provided  $C(\epsilon + \epsilon^{p-1}) \leq \frac{1}{20} \sigma^2 \min(1, \sigma)$ .

Note that we used Sobolev embedding and Gagliardo-Nirenberg's inequality to estimate  $\|z\|_{L_t^\infty L_x^2}$  which states that for any KG admissible pair  $(q, r)$ , one has the following estimate:

$$\|z\|_{L_t^q W_x^{r, 1-(d/2-2/q-d/r)-1/q-2/(dq)}} \leq M_2(m) \quad (60)$$

### D. Estimating $M_1(\tilde{m})$

In order to estimate  $M_1$ , we will use the fact that if  $h = \tilde{h}$  and  $\langle \sigma f_1 + f_2, \psi \rangle = 0$ , then  $2\langle f_1, \psi \rangle + \tilde{h} = a(0) - \frac{a'(0)}{\sigma}$ .  $M_1(\tilde{m})$  has two components. First, we estimate

$$\begin{aligned} \sup_t |\tilde{a}(t)| &\leq \frac{1}{2} (2|\langle f_1, \psi \rangle| + |\tilde{h}|) + \frac{1}{2\sigma} \sup_t \int_0^t e^{-\sigma(t-s)} |\langle F(m(s)), \psi \rangle| ds \\ &\quad + \frac{1}{2\sigma} \sup_t \int_t^\infty e^{\sigma(t-s)} |\langle F(m(s)), \psi \rangle| ds \end{aligned}$$

From (59) and the estimates for  $M_0(\tilde{m})$ , it follows

$$\begin{aligned} \sup_t |\tilde{a}(t)| &\leq \delta\epsilon + \frac{\epsilon}{10} + \frac{1}{\sigma^2} \sup_s |\langle F(m(s)), \psi \rangle| \\ &\leq \delta\epsilon + \frac{\epsilon}{10} + \frac{C}{\sigma^2} (M_1(m)^2 + M_2(m)^2 + M_1(m)^p + M_2(m)^p) \leq \epsilon \end{aligned}$$

provided  $\delta < 1/2$  and  $2C(\epsilon + \epsilon^{p-1}) \leq \sigma^2/4$ .

For the second component, we use Hausdorff-Young's inequality

$$\begin{aligned}\|\tilde{a}\|_{L_t^3} &\leq (\|f_1\|_{L^2} + |\tilde{h}|) \left( \int_0^\infty e^{-3\sigma t} dt \right)^{1/3} + \frac{1}{2\sigma} \|e^{-\sigma|\cdot|}\|_{L^1} \|\langle F(m(s)), \psi \rangle\|_{L_s^3} \\ &\leq (\delta\epsilon + \frac{\epsilon}{10} \min(1, \sigma)) \frac{1}{\min(1, \sigma)} + \frac{1}{2\sigma^2} \left( \int_0^\infty |\langle F(m(s)), \psi \rangle|^3 ds \right)^{1/3}\end{aligned}$$

From Proposition 3 in Ref. 26, we estimate

$$|\langle F(m(s)), \psi \rangle| \leq C(|a(s)|^2 + \|\mathbf{z}(s, \cdot)\|_{L_x^2}^2 + |a(t)|^p + \|\mathbf{z}(s, \cdot)\|_{L_x^p}^p) \quad (61)$$

It follows that

$$\left( \int_0^\infty |\langle F(m(s)), \psi \rangle|^3 ds \right)^{1/3} \leq C(\|a\|_{L_t^6}^2 + \|a\|_{L_t^{3p}}^p + \|\mathbf{z}\|_{L_t^6 L_x^6}^2 + \|\mathbf{z}\|_{L_t^{3p} L_x^6}^p) \quad (62)$$

Since  $p \geq 5$ , we estimate  $\|a\|_{L^6}$ , and  $\|a\|_{L^{3p}}$ . By Gagliardo-Nirenberg's inequality (or log-convexity of  $L^p$  norms), for  $w \geq 3$ ,

$$\|a\|_{L^w(0, \infty)} \leq M_1(m). \quad (63)$$

This follows from

$$\|a\|_{L^w(0, \infty)} \leq \|a\|_{L^3(0, \infty)}^{3/w} \|a\|_{L^\infty(0, \infty)}^{1-3/w} \leq M_1(m). \quad (64)$$

Thus we have

$$\|a\|_{L_t^6}, \|a\|_{L_t^{3p}} \leq M_1(m) \leq \epsilon \quad (65)$$

and because (6, 6), (3p, 6) are KG admissible, it follows that

$$\|\mathbf{z}\|_{L_t^6 L_x^6}, \|\mathbf{z}\|_{L_t^{3p} L_x^6} \leq M_2(m) \leq \epsilon \quad (66)$$

Thus we have

$$\|\tilde{a}\|_{L_t^3} \leq \frac{1}{\min(1, \sigma)} (\delta\epsilon + \min(1, \sigma) \frac{\epsilon}{10}) + C_\sigma (2\epsilon^2 + 2\epsilon^p) \quad (67)$$

and it suffices to require that  $\delta < \min(1, \sigma)/2$  and  $2C_\sigma(\epsilon + \epsilon^{p-1}) \leq \epsilon/4$  in order to conclude that

$$M_1(\tilde{m}) = \max(\|\tilde{a}\|_{L_t^\infty}, \|\tilde{a}\|_{L_t^3}) \leq \epsilon \quad (68)$$

## E. Estimating $M_2(\tilde{m})$

$M_2$  has two components. Firstly, we will estimate

$$\begin{aligned}\|\tilde{z}\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} &\leq C \|\cos(t\sqrt{\mathcal{H}}) P_{a.c.} f_1\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} + \left\| \frac{\sin(t\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}} P_{a.c.} f_2 \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \\ &\quad + \left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}} P_{a.c.} F(m(s)) \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1}\end{aligned}$$

Using Strichartz Estimates and Sobolev Embedding,

$$\|\cos(t\sqrt{\mathcal{H}}) P_{a.c.} f_1\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \leq C \|f_1\|_{H^1} \quad (69)$$

Similarly we get

$$\left\| \frac{\sin(t\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}} P_{a.c.} f_2 \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \leq C \|f_2\|_{L^2} \quad (70)$$

Using (58), we get

$$\begin{aligned} & \left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}} P_{a.c.} F(m(s)) ds \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \\ & \lesssim \left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}} P_{a.c.} \phi^{p-2} (|a(s)|^2 \psi^2 + |\mathbf{z}(s, \cdot)|^2) + |a(s)|^p \psi^p + |\mathbf{z}(s, \cdot)|^p ds \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \\ & \lesssim \|a\|_{L_t^4}^2 + \|a\|_{L_t^p}^2 + \|\mathbf{z}\|_{L_x^\infty(\langle x \rangle^{-3/2}) L_t^2}^2 + \|\mathbf{z}\|_{L_t^p L_x^{2p}}^p \end{aligned}$$

We use Lemma 2.1 and Cauchy-Schwartz Inequality to get  $\|a\|_{L_t^4}$  and  $\|\mathbf{z}\|_{L_x^\infty(\langle x \rangle^{-3/2}) L_t^2}$ . We have

$$\begin{aligned} \left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}} P_{a.c.} \phi^{p-2} |a(s)|^2 \psi^2 ds \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} & \lesssim \|\phi^{p-2} |a(t)|^2 \psi^2\|_{L_t^2 L_x^2(\mathbb{R}; \langle x \rangle^5)} \\ & \lesssim \|a\|_{L_t^4}^2 \end{aligned}$$

Similarly we have

$$\begin{aligned} \left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}} P_{a.c.} \phi^{p-2} |\mathbf{z}(s, \cdot)|^2 ds \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} & \lesssim \|\phi^{p-2} |\mathbf{z}(t, x)|^2\|_{L_t^2 L_x^2(\mathbb{R}; \langle x \rangle^5)} \\ & \lesssim \|\mathbf{z}\|_{L_x^\infty(\langle x \rangle^{-3/2}) L_t^2}^2 \end{aligned}$$

We apply Lemma 3.2 in order to get  $\|a\|_{L_t^p}^2$  and  $\|\mathbf{z}\|_{L_t^p L_x^{2p}}^p$ . We take  $q'_1 = 1$  and  $r'_1 = 2$ .

$$\left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}} P_{a.c.} |a(s)|^p \psi^p + |\mathbf{z}(s, \cdot)|^p ds \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \lesssim \|a\|_{L_t^p}^2 + \|\mathbf{z}\|_{L_t^p L_x^{2p}}^p$$

Thus we have

$$\|\tilde{z}\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \leq C(\|(f_1, f_2)\|_{H^1(R) \times L^2(R)} + \|a\|_{L_t^4}^2 + \|a\|_{L_t^p}^2 + \|\mathbf{z}\|_{L_x^\infty(\langle x \rangle^{-3/2} dx) L_t^2}^2 + \|\mathbf{z}\|_{L_t^p L_x^{2p}}^p)$$

Since  $p \geq 5$ , we have

$$\|a\|_{L_t^4}, \|a\|_{L_t^p} \leq M_1(m) \leq \epsilon \quad (71)$$

From Strichartz Estimates,  $p \geq 5$  implies that  $\frac{2}{p} + \frac{1}{2p} \leq \frac{1}{2}$ , thus we get

$$\|\mathbf{z}\|_{L_t^p L_x^{2p}} \leq M_2(m) \quad (72)$$

Also it is clear that  $\|\mathbf{z}\|_{L_x^\infty(\langle x \rangle^{-3/2} dx) L_t^2} \leq M_2(m) \leq \epsilon$ .

It follows that

$$\|\tilde{z}\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \leq C_1(\delta\epsilon + 2\epsilon^2 + 2\epsilon^p) \leq \epsilon \quad (73)$$

if  $C_1\delta \leq 1/4$ ,  $C_1(\epsilon + \epsilon^{p-1}) \leq 1/4$ .

For the second component

$$\begin{aligned} \|\tilde{z}\|_{L_x^\infty(\langle x \rangle^{-3/2} dx) L_t^2} & \leq C \left\| \cos(t\sqrt{\mathcal{H}}) P_{a.c.} f_1 \right\|_{L_x^\infty(\langle x \rangle^{-3/2} dx) L_t^2} \\ & + \left\| \frac{\sin(t\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}} P_{a.c.} f_2 \right\|_{L_x^\infty(\langle x \rangle^{-3/2} dx) L_t^2} \\ & + \left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}} P_{a.c.} F(m(s)) ds \right\|_{L_x^\infty(\langle x \rangle^{-3/2} dx) L_t^2} \end{aligned}$$



By Lemma (2.3), we have

$$\|\cos(t\sqrt{\mathcal{H}})P_{a.c.}f_1\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \leq C\|f_1\|_{L^2} \quad (74)$$

and

$$\left\|\frac{\sin(t\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}}P_{a.c.}f_2\right\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \leq C\|f_2\|_{H^1} \quad (75)$$

In order to estimate the last term, we will use

$$\left\|\int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}}P_{a.c.}F(m(s))ds\right\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \lesssim \begin{cases} \|F\|_{L_t^1L_x^2} \\ \|F\|_{L_t^2L_x^2(\mathbb{R};\langle x\rangle^3dx)} \end{cases} \quad (76)$$

The first inequality follows from Lemma 2.3.

$$\begin{aligned} & \left\|\int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}}P_{a.c.}F(m(s))ds\right\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \\ & \leq \int_0^t dt \left\|\langle x\rangle^{-3/2}e^{it\sqrt{\mathcal{H}}}\left(\frac{e^{-is\sqrt{\mathcal{H}}}}{\sqrt{\mathcal{H}}}P_{a.c.}F(m(s))\right)\right\|_{L_x^\infty L_t^2} \\ & \lesssim \int_0^\infty ds \left\|\frac{e^{-is\sqrt{\mathcal{H}}}}{\sqrt{\mathcal{H}}}P_{a.c.}F(m(s))\right\|_{L_x^2} \lesssim \|F\|_{L_t^1L_x^2} \end{aligned}$$

The second inequality follows from Lemma 2.2 and Cauchy-Schwartz Inequality.

$$\left\|\int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}}P_{a.c.}F(m(s))ds\right\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \lesssim \|\langle x\rangle^{3/2}F\|_{L_t^1L_x^2} \lesssim \|F\|_{L_t^2L_x^2(\mathbb{R};\langle x\rangle^3dx)}$$

Using (58) and (76), we get

$$\begin{aligned} & \left\|\int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}}P_{a.c.}F(m(s))ds\right\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \\ & \lesssim \|\phi^{p-2}|a(t)|^2\psi^2\|_{L_t^2L_x^2(\mathbb{R};\langle x\rangle^3)} + \|\phi^{p-2}|z(t,x)|^2\|_{L_t^2L_x^2(\mathbb{R};\langle x\rangle^3)} \\ & \quad + \| |a(t)|^p\psi^p\|_{L_t^2L_x^2(\mathbb{R};\langle x\rangle^3)} + \| |z(t,x)|^p\|_{L_t^1L_x^2} \\ & \lesssim \|a\|_{L_t^4}^2 + \|z\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2}^2 + \|a\|_{L_t^{2p}}^p + \|z\|_{L_t^pL_x^{2p}}^p \end{aligned}$$

Since  $p \geq 5$ , we can control  $\|a\|_{L_t^4}$ ,  $\|z\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2}$ ,  $\|a\|_{L_t^{2p}}$ ,  $\|z\|_{L_t^pL_x^{2p}}$ . It follows that

$$\|z\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \leq C_2(\delta\epsilon + 2\epsilon^2 + 2\epsilon^p) \leq \epsilon \quad (77)$$

if  $C_2\delta \leq 1/4$ ,  $C_2(\epsilon + \epsilon^{p-1}) \leq 1/4$ . Thus we can conclude

$$M_2(\tilde{m}) = \max(\|z\|_{L_t^5L_x^{10}\cap L_t^\infty H_x^1}, \|z\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2}) \leq \epsilon \quad (78)$$

Thus we can say that for appropriately chosen  $\epsilon$  and  $\delta$ , so that  $\|(f_1, f_2)\|_{H^1 \times L^2}$ , we can establish  $\Lambda: B_x(\epsilon) \rightarrow B_x(\epsilon)$ . Note that all the estimates leading to that conclusion were in the form

$$\|\Lambda(m)\|_X \leq C\|m\|_X(1 + \|m_1\|_X + \|m_2\|_X)^{p-1}. \quad (79)$$

In order to finish the proof of the contraction mapping theorem, we have to prove that  $\Lambda$  is a contraction, i.e.,  $\|\Lambda(m_1) - \Lambda(m_2)\| \leq C\|m_2 - m_1\|$  for some  $C < 1$ . It is standard in this line of reasoning that if one has (79) and the non-linearity  $F$  has some “multilinear” feature, then the proof of (79) can be used to show the contraction of the same map. Indeed, all we have to observe that,

similar to (58), we have

$$|F(a, \mathbf{z}) - F(b, \mathbf{w})| \leq C_{p,\phi,\psi}(\phi^{p-2}[(|a-b|)(|a|+|b|) + |\mathbf{z}-\mathbf{w}|(|\mathbf{z}|+|\mathbf{w}|)] \\ + \psi^p|a-b|(|a|+|b|)^{p-1} + |\mathbf{z}-\mathbf{w}|(|\mathbf{z}|+|\mathbf{w}|)^{p-1}). \quad (80)$$

This last estimate will allow us to do the same estimates as before, except that the entries will be the difference term  $m_1 - m_2$ . This way, we show the following analogue of (79)

$$\|\Lambda(m_1) - \Lambda(m_2)\|_X \leq C\|m_1 - m_2\|_X(\|m_1\|_X + \|m_2\|_X)(1 + \|m_1\|_X + \|m_2\|_X)^{p-2},$$

which implies the desired contractivity of the map  $\Lambda$  for small  $\|m_1\|_X, \|m_2\|_X$ .

## ACKNOWLEDGMENTS

Stanislavova supported in part by NSF-DMS 0807894.

## APPENDIX A: PROOF OF LEMMA 1.3

We use the theory developed in the Titchmarsh book,<sup>28</sup> The non-resonant condition is equivalent to  $W_{\mathcal{H}}(1) \neq 0$ , where  $W_{\mathcal{H}}(\lambda) = f'_1(x, \lambda)f_2(x, \lambda) - f_1(x, \lambda)f'_2(x, \lambda)$  was defined in Sec. III C. In our situation, we have  $\mathcal{H} = 1 - \partial_x^2 - \frac{p(p+1)}{2} \cosh^{-2}(\frac{(p-1)}{2}x) = 1 + \mathcal{L}$ . Therefore  $W_{\mathcal{H}}(\lambda) = W_{\mathcal{L}}(\lambda - 1)$  and hence, we need to show  $W_{\mathcal{L}}(0) \neq 0$ .

In order to prove that claim, we use the results in Secs. 2.18 and 4.19 in Ref. 28. Namely, take  $\theta, \phi$ , which solve  $(\mathcal{L} - \lambda)y = 0$ , subject to  $\phi(0) = 0, \phi'(0) = -1$ , while  $\theta(0) = 1, \theta'(0) = 0$ . Then, if one represents

$$f_1(x, \lambda) = \theta(x, \lambda) + m_1(\lambda)\phi(x, \lambda) \\ f_2(x, \lambda) = \theta(x, \lambda) + m_2(\lambda)\phi(x, \lambda),$$

we have  $W_{\mathcal{L}}(\lambda) = m_1(\lambda) - m_2(\lambda)$ .

In order to reduce to the situation in Sec. 4.19 in the book of Titchmarsh, we need a change of variables. The equation that we need to consider is  $(\mathcal{L} - \lambda)f = 0$ , which is

$$f''(x) + \frac{p(p+1)}{2} \cosh^{-2}(\frac{(p-1)}{2}x)f(x) = -\lambda f(x)$$

Changing variables  $x \rightarrow (p-1)^{-1}y$  yields the following equation

$$g''(y) + \frac{2p(p+1)}{(p-1)^2} \frac{1}{4 \cosh^2(\frac{y}{2})} g(y) = -\lambda g(y).$$

Solving the quadratic equation

$$\alpha(1-\alpha) = -\frac{2p(p+1)}{(p-1)^2}$$

which has solutions  $\alpha = 2p/(p-1), \alpha = -(p+1)/(p-1)$  puts the equation for  $g$  in the form

$$g''(y) + \left( \lambda - \frac{\alpha(1-\alpha)}{4 \cosh^2(\frac{y}{2})} \right) g(y) = 0$$

which is the form of (4.18.5) in Ref. 28. It is then argued in Ref. 28, that by the fact that since the potential  $\cosh^{-2}(\frac{y}{2})$  is even, we have  $m_1(\lambda) = -m_2(\lambda)$  and

$$m_2(\lambda) = \frac{\Gamma(1 - \frac{\alpha}{2} - i\sqrt{\lambda})\Gamma(\frac{1}{2} + \frac{\alpha}{2} - i\sqrt{\lambda})}{\Gamma(\frac{1}{2} - \frac{\alpha}{2} - i\sqrt{\lambda})\Gamma(\frac{\alpha}{2} - i\sqrt{\lambda})},$$

where all  $\Gamma$  functions involved are considered as meromorphic functions on the corresponding domain. Thus,  $\lambda = 0$  is a point of resonance, if and only if the function  $m_2(\lambda)$  has a zero at zero. Observing that the numerator is a product of two  $\Gamma$  functions and is never zero, it remains to show

that for  $\alpha_p = 2p/(p - 1)$ , the point  $\lambda = 0$  is not a pole for the denominator. That is, we have to check that for each integer  $j \geq 0$

$$\frac{1 - \alpha}{2} \neq -j; \quad \frac{\alpha}{2} \neq -j.$$

Indeed, the second one is obvious, since  $\alpha_p = 2p/(p - 1) > 0 \geq -j$ . For the first one, the solution to the inequality gives

$$p \neq 1 + \frac{2}{j - 1}$$

which is certainly satisfied for  $p > 3$ , whenever  $j$  is an integer,  $j \geq 0$ . Note however that  $p = 3, j = 2$  is a solution and therefore, there is a resonance at  $p = 3$ .

- <sup>1</sup>Bates P. and Jones C., "Invariant manifolds for semi-linear partial differential equations," *Dyn. Rep.*, **2**, 1–38.
- <sup>2</sup>Bates P. and Jones C., "The solutions of the nonlinear Klein-Gordon equation near a steady state," *Advanced topics in the theory of dynamical systems* (1989), 1–9.
- <sup>3</sup>Beceanu M., "A critical centre-stable manifold for the cubic focusing Schrödinger equation in three dimensions," Ph.D. dissertation, University of Chicago, 2009.
- <sup>4</sup>Beceanu M., "A centre-stable manifold for the focussing cubic NLS in  $R^{1+3}$ ," *Comm. Math. Phys.* **280**(1), 145205 (2008).
- <sup>5</sup>Buslaev V. S. and Perelman G. S., "Scattering for the nonlinear Schrödinger equation: States that are close to a soliton," *St. Petersburg Math. J.* **4**, 11111142 (1993).
- <sup>6</sup>Buslaev V. S., Perelman G. S., *On the Stability of Solitary Waves for Nonlinear Schrödinger Equations*, American Mathematics Society Translation Series 2, Vol. 164 (Amer. Math. Soc., Providence, RI, 1995), pp. 7598.
- <sup>7</sup>Buslaev V. S. and Sulem C., "On asymptotic stability of solitary waves for nonlinear Schrödinger equations," *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20**, 419475 (2003), p. 1.
- <sup>8</sup>Cazenave T., *Semilinear Schrödinger Equations* (AMS, Providence, RI, 2003).
- <sup>9</sup>Chang S., Gustafson S., Nakanishi K., and Tsai T., "Spectra of Linearized Operators for NLS Solitary Waves," *SIAM J. Math. Anal.* **39**, 1070–1111 (2003).
- <sup>10</sup>Gesztesy F., Jones C. K. R. T., Latushkin Y., and Stanislavova M., "A Spectral Mapping Theorem and Invariant Manifolds for Nonlinear Schrödinger Equations," *Indiana Univ. Math. J.* **49**, 221–243 (2000).
- <sup>11</sup>Goldberg M. and Schlag W., "Dispersive estimates for Schrödinger operators in dimensions one and three," *Commun. Math. Phys.* **251**(1), 157178 (2004).
- <sup>12</sup>Christ M. and Kiselev A., "Maximal functions associated with filtrations," *J. Funct. Anal.* **179**, 409–425 (2001).
- <sup>13</sup>Deift P. and Trubowitz E., "Inverse scattering on the line," *Commun. Pure Appl. Math.* **32**, 121–251 (1979).
- <sup>14</sup>Karageorgis P. and Strauss W., "Instability of steady states for nonlinear wave and heat equations," *J. Differential Equations* **241**(1), 184–205 (2007).
- <sup>15</sup>Krieger J. and Schlag W., "Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension," *J. Am. Math. Soc.* **19**(4), 815920 (2006).
- <sup>16</sup>Krieger J. and Schlag W., "On the focusing critical semi-linear wave equation," *Am. J. Math.* **129**(3), 843–913 (2007).
- <sup>17</sup>Kwong M., "Uniqueness of positive solutions of  $\Delta u - u + u^p = 0$  in  $R^n$ ," *Arch. Rational Mech. Anal.* **105**(3), 243–266 (1989).
- <sup>18</sup>Mizumachi T., "Asymptotic stability of small solitary waves to 1D nonlinear Schrödinger equations with potential," *J. Math. Kyoto Univ.* **48**(3), 471–497 (2008).
- <sup>19</sup>Nakamura M. and Ozawa T., "The Cauchy problem for nonlinear Klein-Gordon equations in the Sobolev spaces," *Publ. Res. Inst. Math. Sci.* **37**(3), 255–293 (2001).
- <sup>20</sup>Pillet C. A. and Wayne C. E., "Invariant manifolds for a class of dispersive, Hamiltonian, partial differential equations," *J. Differential Equations* **141**, 310326 (1997).
- <sup>21</sup>Rodnianski I., Schlag W., and Soffer A., "Asymptotic stability of N-soliton states of NLS," <http://arxiv.org/list/math/0309114>, 2003.
- <sup>22</sup>Soffer A. and Weinstein M., "Multichannel nonlinear scattering for non-integrable equations," *Commun. Math. Phys.* **133**, 119146 (1990).
- <sup>23</sup>Soffer A. and Weinstein M., "Multichannel nonlinear scattering. 2. The case of anisotropic potentials and data," *J. Differential Equations* **98**, 376390 (1992).
- <sup>24</sup>Soffer A. and Weinstein M., "Selection of the ground state for nonlinear Schrödinger equations," *Rev. Math. Phys.* **168**, 9771071 (2004).
- <sup>25</sup>Schlag W., "Stable manifolds for an orbitally unstable nonlinear Schrödinger equation," *Ann. Math.* **169**(1), 139–227 (2009).
- <sup>26</sup>Stanislavova M., "A. Stefanov, On Precise Center Stable Manifold Theorems for Certain Reaction-Diffusion and Klein-Gordon Equations," *Phys. D* **238**, 2298–2307 (2009).
- <sup>27</sup>Smith H. F. and Sogge C. D., "Global Strichartz estimates for nontrapping perturbations of the Laplacian," *Commun. Partial Differ. Equ.* **25**, 2171–2183 (2000).
- <sup>28</sup>Titchmarsh E. C., *Eigenfunction Expansions Associated with Second-Order Differential Equations. Part I*, 2nd ed. (Clarendon, Oxford, 1962).

- <sup>29</sup> Weder R., “ $L^p$ - $L^{\tilde{p}}$  estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential,” [J. Funct. Anal.](#) **170**(1), 3768 (2000).
- <sup>30</sup> Weder R., “ $W_{k,p}$ - $W_{k,p}$ -continuity of the Schrödinger wave operators on the line,” [Commun. Math. Phys.](#) **208**(2), 507520 (1999).
- <sup>31</sup> Weinstein M., “Modulational stability of ground states of nonlinear Schrödinger equations,” [SIAM J. Math. Anal.](#) **16**(3), 472–491 (1985).
- <sup>32</sup> Weinstein M., “Lyapunov stability of ground states of nonlinear dispersive evolution equations,” [Commun. Pure Appl. Math.](#) **39**(1), 51–67 (1986).